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Fourier Transform

• decomposes a function into its sine and/or cosine parts representing the frequency spectrum of the original function.
• takes a complex-valued function f to a complex-valued function defined by:

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx.$$

• the real parts of the resulting complex-valued function represent the amplitudes of their respective frequencies, while the imaginary parts represent the phase shifts.

Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx}. \quad \text{where} \quad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx.$$

Since: $e^{inx} = \cos(nx) + i\sin(nx)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(n)\right], \quad \text{where}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Discrete Fourier Transform (DFT)

Computers work with discrete input/output, so the Discrete Fourier Transform (DFT) must be used:

$$f_j = \sum_{k=0}^{n-1} x_k e^{-\frac{2\pi i}{n} jk} \qquad j = 0, \ldots, n-1.$$

The complex numbers x_0, ..., x_{n-1} are transformed into the complex numbers f_0, ..., f_{n-1}
Evaluating these sums directly would take $O(n^2)$ arithmetical operations. A Fast Fourier Transform (FFT) is an algorithm to compute the same result in only O(n log n) operations. By far the most common FFT is the Cooley-Tukey algorithm.

Discrete Cosine Transform (DCT)

When the input data contains only real numbers from an even (ie symmetric) function, the sin component is 0 and the DFT becomes a DCT. There are 4 variants, however.

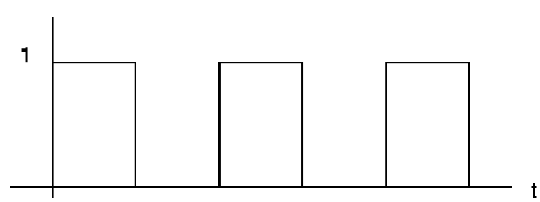
DCT Type II (used in JPEG – repeated for a 2D transform)

$$f_j = \sum_{k=0}^{n-1} x_k \cos\left[\frac{\pi}{n} j(k+1/2)\right]$$

DCT Type IV (used in mp3 – a variant called MDCT)

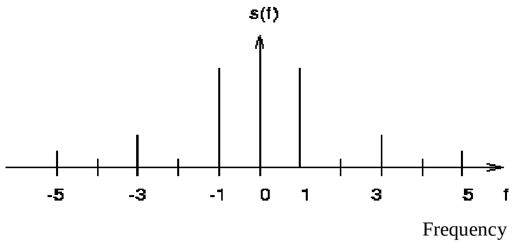
$$f_j = \sum_{k=0}^{n-1} x_k \cos\left[\frac{\pi}{n}(j+1/2)(k+1/2)\right]$$

Pictorial representation

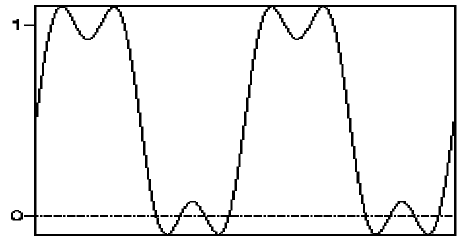


A Square wave

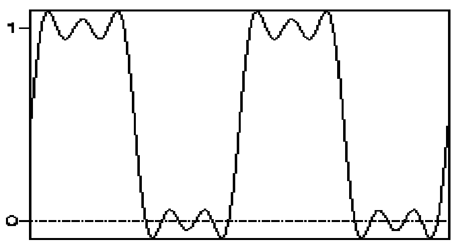
Square wave spectrum



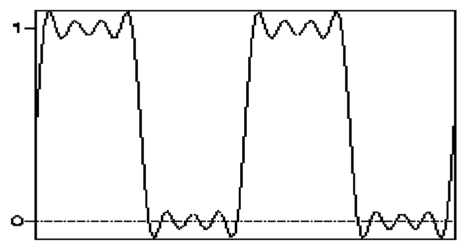
Two sin waves



Three sin waves



Four sin waves



Coefficient coding

- Run length encoding
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0 = 10 x 0
- Differences
  – 5, 4, 5, 6, 8, 6, 4, 5 = 5, -1, 1, 1, 2, -2, -2, 1
- Huffman coding
  – replacing a set of values of fixed size code words with an optimal set of different sized code words based on the statistics of the input data
  – frequency distribution of the symbols is constructed
  – compressed representation for each symbol is then decided

Dictionary approach

- look at the data ***as it arrives*** and form a dictionary on the fly
- As the dictionary is formed, it can be used to look up new input, dynamically
- if the new input existed earlier in the stream, the dictionary position can be transmitted instead of the new input codes
- known as "substitutional" compression algorithms

(J Ziv and A Lempel in the 1970s)